Fourier Transformation

Recall: For a periodic function $f(t)$, it has a Fourier series, with period T

$$f(t) = \sum_{n \in \mathbb{N}} c_n e^{in\omega_0 t}$$

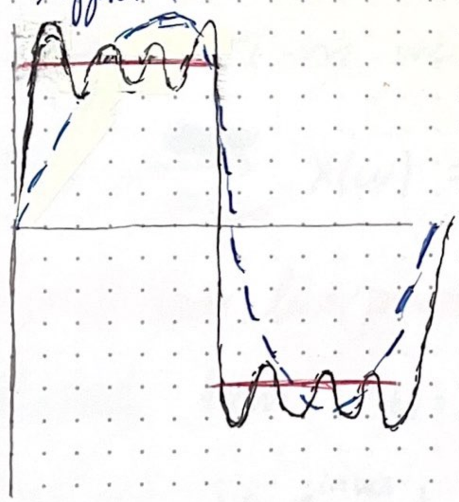
where $\omega_0 = \frac{2\pi}{T}$ is the fundamental frequency,

$$c_n = \frac{1}{T} \int_T f(t) e^{-in\omega_0 t} \, dt$$

Motivation for FS?

↳ Historically: Solutions for heat equation $\quad \frac{\partial u}{\partial t} = \Delta u$ ↳ derivatives of trigonometric nice

↳ Approximate functions by sine + cosines



↳ Behaviour is nicer and well understood.

From lecture (beginning), derived formula for c_n

Sketch: • Every function written as even+odd $\quad f(t) = \frac{1}{2}(f(t) - f(-t)) + \frac{1}{2}(f(t) + f(-t))$

• Derive formula for coeff. a_n, b_n resp. for of even, odd parts
↳ multiply by $\cos(m\omega_0 t), \sin(m\omega_0 t)$, trig identities + $\perp$ of cos, sin

• Exponential representation of cos/sin, combine. "

__Issue:__ FS only works for __periodic__ functions.

For f a non periodic function, we can consider it having a period at $T = \infty$.

As $T \to \infty$, $n\omega_0 = \frac{2\pi}{T}$ now can take on any value, as ω_0 small and $n \in \mathbb{N}$ arbitrary; so let $\omega = n\omega_0$ a new variable, and $\omega_0 = \Delta\omega$

With $Tc_n = \int_T f(t)\, e^{-i\overbrace{n\omega_0}^{\omega} t}\, dt$, Tc_n is no longer discrete but continuous, call it $X(\omega)$.

Thus, as $T \to \infty$ we set

$$Tc_n \xleftarrow[T\to\infty]{} X(\omega) = \int_{\mathbb{R}} f(t)\, e^{-i\omega t}\, dt$$

Recall from list/previous session → convergence holds in $L^2(\mathbb{R})$

Similarly, from $f(t) = \sum_{n\in\mathbb{N}} c_n e^{in\omega_0 t}$

$$\Leftrightarrow f(t) = \sum_{n\in\mathbb{N}} Tc_n e^{in\omega_0 t} \frac{1}{T} = \sum_{n\in\mathbb{N}} Tc_n \frac{\omega_0}{2\pi} e^{in\omega_0 t}$$

$$\Leftrightarrow f(t) = \frac{1}{2\pi} \sum_{n\in\mathbb{N}} Tc_n e^{in\omega_0 t} \Delta\omega$$

$$\overset{T\to\infty}{=} \frac{1}{2\pi} \int_{\mathbb{R}} X(\omega)\, e^{i\omega t}\, d\omega$$

Meaning: • We can extend the notion/idea of FS to non-periodic functions as considering them as "$T=\infty$ periodic".

• The FT transforms a function of time to a function of frequency, and we can reverse it with the inverse FT.

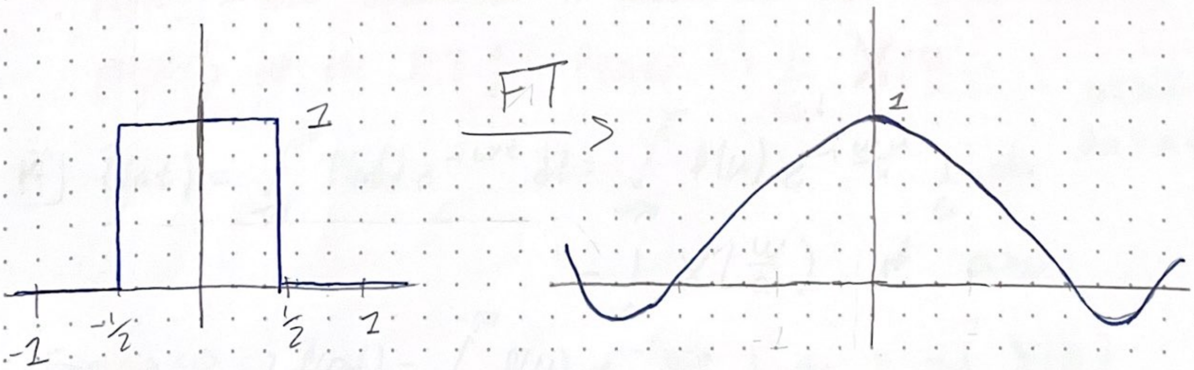
$\boxed{\text{Ex}}$ i) Take a box function $f(x) = \begin{cases} 1 & |x| \le \frac{1}{2} \\ 0 & \text{else} \end{cases}$

$$\Rightarrow X(\omega) = \hat{f}(t) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \underbrace{f(t)}_{=1} e^{-i\omega t} = \frac{e^{-i\omega t}}{-i\omega} \Bigg|_{t=-\frac{1}{2}}^{\frac{1}{2}}$$

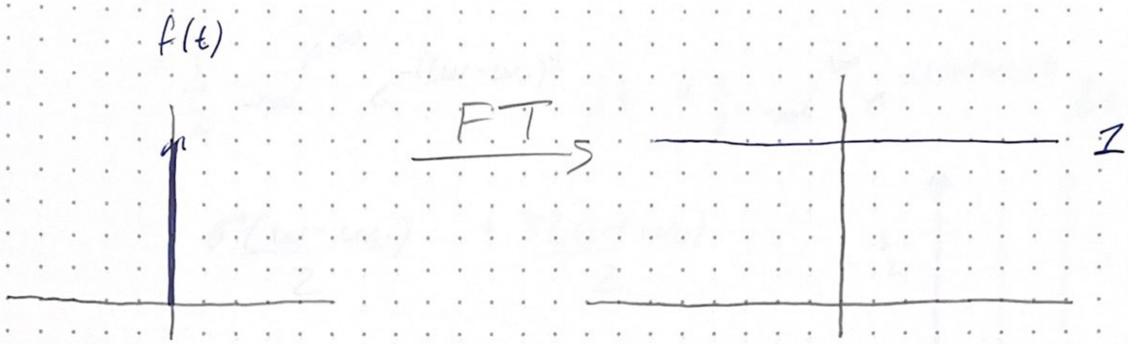
$$= \frac{e^{i\omega/2} - e^{-i\omega/2}}{i\omega} = \frac{2\sin(\omega/2)}{\omega} \cancel{= \operatorname{sinc}\left(\frac{\omega}{2}\right)}$$

$$= \operatorname{sinc}\left(\frac{\omega}{2}\right)$$



ii] $f(t) = \delta(t)$ (Dirac Delta)

$$X(\omega) = \hat{\delta}(t) = \int_{-\infty}^{\infty} \delta(t)\, e^{-i\omega t}\, dt = e^{-i\omega 0} = 1$$



This is a limiting case of i, as the width of the box impulse $\to 0$, the FT sinc($\bullet$) becomes infinitely wider. The converse also holds, and can be shown via inverse FT. box wider $\to$ FT narrower.

Why? Think of sound; stretching a sound over time will make it move more slowly $\Rightarrow$ frequencies get shifted lower. Conservation of energy means its amplitude (height) gets scaled up.

! Note: Above behaviour is known as the "time scaling" property of the FT; $f(at) \xrightarrow{FT} \frac{1}{|a|} X(\frac{\omega}{a})$

$u = at$
$du = a\,dt$

Pf] $\hat{f}(at) = \int_{-\infty}^{\infty} f(at)\, e^{-i\omega t}\, dt = \int_{-\infty}^{\infty} f(u)\, e^{-i\frac{\omega u}{a}} \frac{1}{a}\, du$

$$= \frac{1}{a} X\left(\frac{\omega}{a}\right) \text{ if } a > 0$$

For $a < 0 \Rightarrow \hat{f}(at) = \int_{\infty}^{-\infty} f(u)\, e^{-i\frac{\omega u}{a}} \frac{1}{a}\, du = -\frac{1}{a} X\left(\frac{\omega}{a}\right)$

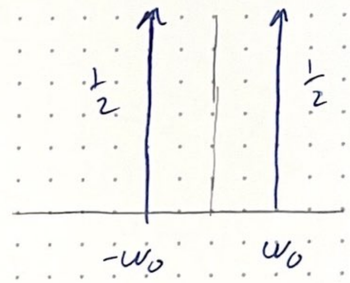
Thus $\hat{f}(at) = \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$

iii] $f(t) = \cos(\omega_0 t)$

$$X(\omega) = \frac{1}{2}\int_{-\infty}^{\infty}\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right) e^{-ij\omega t}\,dt$$

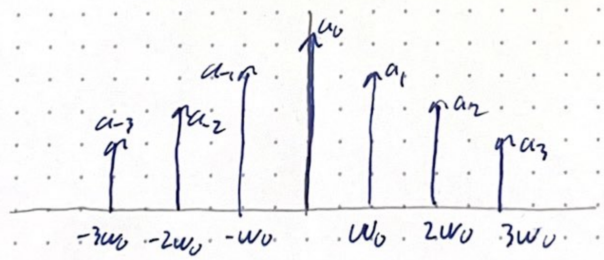
$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{-i(\omega-\omega_0)t}\,dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{-i(\omega+\omega_0)t}\,dt$$

$$= \frac{\delta(\omega-\omega_0)}{2} + \frac{\delta(\omega+\omega_0)}{2}$$



In general, for $f(t) = \sum_{n\in\mathbb{N}} a_n e^{in\omega_0 t}$

$$X(\omega) = \sum_{n\in\mathbb{N}} a_n \int_{-\infty}^{\infty} e^{i(n\omega_0-\omega)t}\,dt = \sum_{n\in\mathbb{N}} a_n\,\delta(\omega - n\omega_0)$$



Applications of FT

- Number Theory => Basel Problem: $\sum_{n\geq 1}\frac{1}{n^2}$ exact sum?

  ↳ $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} x e^{-inx}\,dx = \frac{\cos(n\pi)}{n} i = \frac{(-1)^n}{n} i$

  & Parsevals identity: $\sum_{n\in\mathbb{N}} |c_n|^2 = 2\sum_{n\geq 1}\frac{1}{n^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{\pi^2}{3}$

- Signal processing => Decompose picture / sound into amplitude and phase for analysis

5/5